

$$\dot{x}(t) = f(x(t), u(t)) \quad ; \quad x(0) = x_0$$

$$J(x, u) = \int_0^T \phi(x(\tau), u(\tau)) d\tau$$

$$\text{with } x(t) := \begin{bmatrix} x_1(t) & x_2(t) \end{bmatrix}^T$$

$$x_0 = \begin{bmatrix} 0.05 & 0 \end{bmatrix}^T$$

$$Q = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$; R = 0.2 \quad ; T = 0.78$$

$$f(x(t), u(t)) := \begin{bmatrix} f_1(x(t), u(t)) \\ f_2(x(t)) \end{bmatrix}$$

$$\phi(x(t), u(t)) = \frac{1}{2} x(t)^T Q x(t) + \frac{1}{2} R u^2(t)$$

$$f_1(x, u) := -2(x_1 + 0.25) + (x_2 + 0.5) v(x_1) - (x_1 + 0.25) u$$

$$f_2(x) := 0.5 - x_2 - (x_2 + 0.5) v(x_1)$$

$$v(x_1) := e^{\frac{25x_1}{x_1+2}}$$

$$\left(v'(x_1) = \frac{50}{(x_1+2)^2} v(x_1) \text{ and } v''(x_1) = -100 \frac{(x_1-23)}{(x_1+2)^4} v(x_1) \right)$$

Method 1: Traditional Gradient Descent:

$$\text{Let } H(x(t), u(t), \lambda(t)) = \frac{1}{2} x(t)^T Q x(t) + \frac{1}{2} R u^2(t) + \lambda(t)^T f(x(t), u(t))$$

$$\frac{\partial f}{\partial x}(x, u) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x, u) & \frac{\partial f_1}{\partial x_2}(x, u) \\ \frac{\partial f_2}{\partial x_1}(x, u) & \frac{\partial f_2}{\partial x_2}(x, u) \end{bmatrix} = \begin{bmatrix} -2 + (x_2 + 0.5) v'(x_1) - u & v(x_1) \\ -(x_2 + 0.5) v'(x_1) & -1 - v(x_1) \end{bmatrix}$$

$$\frac{\partial f}{\partial u}(x, u) = \begin{bmatrix} \frac{\partial f_1}{\partial u}(x, u) \\ \frac{\partial f_2}{\partial u}(x, u) \end{bmatrix} = \begin{bmatrix} -(x_1 + 0.25) \\ 0 \end{bmatrix}$$

$$\dot{\lambda}(t) = - \left[\frac{\partial H}{\partial x}(x(t), u(t), \lambda(t)) \right]^T = -Q x(t) - \left[\frac{\partial f}{\partial x}(x(t), u(t)) \right]^T \lambda(t) \quad (\text{Costate Equation})$$

$$\frac{\partial H}{\partial u}(x(t), u(t), \lambda(t)) = R u(t) + \left[\frac{\partial f}{\partial u}(x(t), u(t)) \right]^T \lambda(t) \quad (\text{Gradient})$$

Method 2: Modified Projected Gradient Descent

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$$\begin{aligned}
 \frac{d}{dt} \left(\frac{\partial f}{\partial x}(x, u) \right) &= \frac{d}{dt} \begin{bmatrix} -2 + (x_2 + 0.5) v'(x_1) - u & v(x_1) \\ -(x_2 + 0.5) v'(x_1) & -1 - v(x_1) \end{bmatrix} \\
 &= \begin{bmatrix} \dot{x}_2 v'(x_1) + (x_2 + 0.5) v''(x_1) \dot{x}_1 - \dot{u} & v'(x_1) \dot{x}_1 \\ -\dot{x}_2 v'(x_1) - (x_2 + 0.5) v''(x_1) \dot{x}_1 & -v'(x_1) \dot{x}_1 \end{bmatrix} \\
 &= \begin{bmatrix} f_2(x) v'(x_1) + (x_2 + 0.5) v''(x_1) f_1(x, u) - \dot{u} & v'(x_1) f_1(x, u) \\ -f_2(x) v'(x_1) - (x_2 + 0.5) v''(x_1) f_1(x, u) & -v'(x_1) f_1(x, u) \end{bmatrix} \\
 F(x(t), u(t)) &:= \frac{\partial f}{\partial x}(x(t), u(t)) \underbrace{\left[\frac{\partial \phi}{\partial x}(x(t), u(t)) \right]^T}_{q(t)} + \frac{\partial f}{\partial u}(x(t), u(t)) \underbrace{\left[\frac{\partial \phi}{\partial u}(x(t), u(t)) \right]^T}_{r(t)} - \underbrace{\frac{d}{dt} \left[\frac{\partial \phi}{\partial x}(x(t), u(t)) \right]^T}_{\dot{q}(t)}
 \end{aligned}$$

$$q(t) = Qx(t) \quad ; \quad r(t) = Ru(t)$$

$$\dot{q}(t) = Q\dot{x}(t) = Qf(x(t), u(t))$$

Algorithm for Both Methods:

1. Start with an initial guess $u^0(t)$ and set $i=0$
2. Integrate $\dot{x}^i(t) = f(x^i(t), u^i(t))$ to compute $x^i(t)$ for $0 \leq t \leq T$
3. Compute the Costate variable $\lambda^i(t)$
4. Update control $u^{i+1}(t) = u^i(t) - \alpha (r_i(t) + B_i^T(t) \lambda^i(t))$
5. If $\|r_i(t) + B_i^T(t) \lambda^i(t)\| < \epsilon$, Stop and $\bar{u}(t) = u^{i+1}(t)$
Else $i=i+1$ and go to step 2.

The Difference between the two methods is in step 3.

For Method 1: $\dot{\lambda}^i(t) = -A_i^T(t)\lambda^i(t) - q_i(t)$ $\lambda(T) = 0$

For Method 2:
$$\begin{cases} \dot{P}(t) = [A_i(t) - A_i^T(t)]P(t) - P^2(t) + A_i(t)A_i^T(t) + B_i(t)B_i^T(t) - \dot{A}_i^T(t) \\ \dot{w}(t) = [A_i(t) - A_i^T(t) - P_i(t)]w(t) + F_i(t) \\ \dot{\lambda}^i(t) = P(t)\lambda(t) + w(t) \end{cases}$$

with $P(0) = -A_i^T(0)$

$w(0) = -q_i(0)$

$\lambda^i(T) = 0$

where $A_i(t) := \frac{\partial f}{\partial x}(x^i(t), u^i(t))$

$B_i(t) := \frac{\partial f}{\partial u}(x^i(t), u^i(t))$

$q_i(t) := Qx^i(t)$

$r_i(t) := Ru^i(t)$

$\dot{A}_i(t) = \frac{d}{dt}A_i(t)$

$F_i(t) := A_i(t)q_i(t) + B_i(t)r_i(t) - \dot{q}_i(t)$